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Investigating Some Important Characteristics of Soft Hilbert Space

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Abstract:
Main aim of this work is to discuss some important characteristics of soft Hilbert Space. In this present work we have analysed some important properties like orthogonality, orthonormality in soft Hilbert space. We discussed some examples of soft Hilbert space such as the soft l^2 space and some inequality results on this space have been given. Moreover, it has been demonstrated that each soft Hilbert space contains an orthonormal basis, which can be derived using the Gram–Schmidt orthogonalization procedure within a soft inner product space.
***AMS Classification**: 03E72, 08A72*
***Keywords:** Soft Hilbert Space, Orthogonal, Orthonormal, Soft l^2 space, Orthonormal basis.*

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Introduction

In the year 1999, Molodtsov [16] initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties. He has demonstrated numerous applications of this theory in addressing practical problems across various fields. Research works in soft set theory and its applications in various fields have been progressing rapidly since Maji et al. ([13],[14]) introduced several operations on soft sets and applied it to decision making problems. In the line of reduction and addition of parameters of soft sets some works have been done by Chen [3], Pei and Miao [17] , Kong et al. [12] , Zou and Xiao [19]. Aktas and Cagman [1] introduced the notion of soft group and discussed various properties. Jun ([11]) investigated soft BCK/BCI - algebras and its application in ideal theory. Feng et al. [9] worked on soft semirings, soft ideals and idealistic soft semirings. Ali et al. [2] and Shabir and Irfan Ali ([2]) studied soft semigroups and soft ideals over a semi group which characterize generalized fuzzy ideals and fuzzy ideals with thresholds of a semigroup. The idea of soft topological spaces was first given by M. Shabir, M. Naz [18] and mappings between soft sets were described by P. Majumdar, S. K. Samanta [15]. Feng et al. [10] worked on soft sets combined with fuzzy sets and rough sets.

In ([4],[5]) Das and Samanta have introduced a notion of soft real sets, soft real numbers, soft complex sets, soft complex numbers and some of their basic properties have been investigated. In ([6],[7]) Das and Samanta introduced the concept of `soft metric', `soft linear spaces', `soft norm' on a `soft linear spaces' and studied various properties of `soft metric spaces' and `soft normed linear spaces' in details. In [8] they have introduced a notion of soft inner product on soft linear space and studied some of its properties.

In this work we have studied some significant characteristic of the soft Hilbert space. Properties like orthogonality, orthonormality in soft Hilbert space are investigated. In the second section some preliminary results are mentioned. In the third section, notion of orthogonality, orthonormality in soft Hilbert space are given and its properties are being investigated. Also we discussed some examples of soft Hilbert space such as the soft l^2 space and some inequality type results on this space have been given. Moreover, it has been demonstrated that each soft Hilbert space contains an orthonormal basis, which can be derived using the Gram–Schmidt orthogonalization procedure within a 'soft inner product space'.

Preliminaries

Definition 2.1 (see [8]): Let W be an universe set and E be a set of parameters. Let $P(W)$ be the power set of W and C denotes a non-empty subset of E. A soft set over W is t defined by the pair (F, C), where F define as $F: C \to P(W)$. In essence, a soft set over W can be viewed parametrized collection of subsets of the universe W. For each $\varepsilon \in C$, $F(\varepsilon)$ may be referred as the set of ε- approximate elements of the soft set (F, C) .

Definition 2.2 (see [14]): For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if

(1) $A \subseteq B$ and
(2) for all $e \in A,\ F(e) \subseteq G(e)$
We write $(F,A)\ \tilde{\subseteq}\ (G,B)$. *(F,A)* is said to be a soft super set of (G,B), if (G,B) is a soft subset of (F,A). We denote it by $(F,A)\ \tilde{\supseteq}\ (G,B)$.

Definition 2.3 (see [14]): Two soft set (F,A) and (G,B) over a common universe U are said to be equal if (F,A) is a soft subset of (G,B) and (G,B) is a soft subset of (F,A).

Definition 2.4 (see [14]): The complement of a soft set (F,A) is denoted by $(F,A)^c = (F^c,A)$, where $F^c: A \to P(U)$ is a mapping given by $F^c(e) = U - F(e)$, for all $e \in A$.

Definition 2.5 (see [14]): A soft set (F,A) over U is said to be a NULL soft set denoted by Φ if for all $e \in A, F(e) = \phi$ (null set).

Definition 2.6 (see [14]): A soft set (F,A) over U is said to be an absolute soft set denoted by $\tilde{U}$ if for all $e \in A\,, F(e) = U$.

Definition 2.7 (see [4]): Let U be a non-empty set and A be a non-empty parameter set. Then a function $\varepsilon : A \to U$ is said to be a soft element of U. A soft element ε of U is said to belongs to a soft set B of U , which is denoted by $\varepsilon\ \tilde{\in}\ B$, if $\varepsilon(e) \in B(e), \forall\, e \in A$. Thus for a soft set B of U with respect to the parameter set A, we have $B(e) = \{\, \varepsilon(e): \varepsilon\ \tilde{\in}\ B\}$, $e \in A$.

Definition 2.8 (see [4]): Let $\mathbb{R}$ be the set of real numbers and $\mathcal{B}(\mathbb{R})$ the collection of all non-empty bounded subsets of $\mathbb{R}$ and A taken as a set of parameters. Then a mapping $F: A \to \mathcal{B}(\mathbb{R})$ is called soft real set. It is denoted by (F,A). If specifically (F,A) is a singleton soft set, then after identifying (F,A) with the corresponding soft element, it will be called a soft real number. The set of all soft real numbers is denoted by $\mathbb{R}(A)$.

We use notations $\tilde{x}, \tilde{y}, \tilde{z}$ to denote soft real numbers whereas $\bar{x}\,, \bar{y}, \bar{z}$ will denote a particular type of soft real numbers such that $\bar{x}(\lambda) = x$ for all $\lambda \in A$. For example $\bar{0}$ is the soft real number where $\bar{0}(\lambda) = 0, \forall\, \lambda \in A$.

Definition 2.9 (see [4]): For two soft real numbers $\tilde{r}, \tilde{s}$ we define

(i) $\tilde{r}\ \tilde{\leq}\ \tilde{s}$ if $\tilde{r}(\lambda) \leq \tilde{s}(\lambda)$,for all $\lambda \in A$.

(ii) $\tilde{r}\ \tilde{\geq}\ \tilde{s}$ if $\tilde{r}(\lambda) \geq \tilde{s}(\lambda)$,for all $\lambda \in A$.

(iii) $\tilde{r}\ \tilde{<}\ \tilde{s}$ if $\tilde{r}(\lambda) < \tilde{s}(\lambda)$,for all $\lambda \in A$.

(iv) $\tilde{r}\ \tilde{>}\ \tilde{s}$ if $\tilde{r}(\lambda) > \tilde{s}(\lambda)$,for all $\lambda \in A$.

Definition 2.10 (see [4]): A soft real number $\tilde{r}$ is said to be non-negative if $\tilde{r}(\lambda) \geq 0, \forall\, \lambda \in A$. We denote the set of all non-negative soft real numbers by $\mathbb{R}(A)^*$.

Remark 2.11 (see [7]): Let X be a non-empty set. Let $\check{X}$ be absolute set i.e. $F(\lambda) = X, \forall\, \lambda \in A$, where $(F, A) = \check{X}$. Let $S(\check{X})$ be the collection of the null soft set Φ and those soft sets (F, A) over X for which $F(\lambda) \neq \phi, \forall \lambda \in A$. For $(F, A)(\neq \Phi) \in S(\check{X})$, the collection of all soft elements of (F, A) will be denoted by $SE(F, A)$.

Definition 2.12 (see [7]): Let V be a vector space over a field K and let A be a parameter set. Let (G, A) is a soft set over V. Now G is said to be a soft vector space or soft linear space of V over K if $G(\lambda)$ is a vector space of $V, \forall\, \lambda \in A$.

Definition 2.13 (see [7]): Let G be a soft vector space of V over K. Then a soft element of G is said to be a soft vector of G. In a similar manner a soft element of the soft set (K, A) is said to be a soft scalar, K being the scalar field.

Definition 2.14 (see [8]): Let $\mathbb{C}$ be the set of complex numbers and $\rho(\mathbb{C})$ be the collection of all non-empty bounded subsets of the set of complex numbers. A be a set of parameters. Then a mapping $F: A \to \rho(\mathbb{C})$ is called a soft complex set. It is denoted by (F, A). If in particular (F, A) is a singleton soft set; then identifying (F, A) with the corresponding soft element, it will be called a soft complex number. The set of all soft complex numbers is denoted by $\mathbb{C}(A)$.

Definition 2.15 (see [8]): Let (F, A) be a soft complex set. Then the complex conjugate of (F, A) is denoted by $(\bar{F}, A)$ and is defined by $\bar{F}(\lambda) = \{\bar{z} : z \in F(\lambda)\}$, $\forall\, \lambda \in A$, where $\bar{z}$ is complex conjugate of the ordinary complex number z . The complex conjugate of a soft complex number (F, A) is $\bar{F}(\lambda) = \bar{z}, \in F(\lambda)$, $\forall\, \lambda \in A$.

Definition 2.16 (see [8]): Let (F, A) be a soft complex number. Then the modulus of (F, A) is denoted by $(|F|, A)$ and is defined by $|F|(\lambda) = |z|; z \in F(\lambda), \forall\, \lambda \in A$, where z is an ordinary complex number. Since the modulus of each ordinary complex number is a non-negative real number and by definition of soft real numbers it follows that $(|F|, A)$ is a non-negative soft real number for every soft complex number (F, A).

Definition 2.17 (see [8]): Let $\check{X}$ be the absolute soft vector space i.e., $\check{X}(\lambda) = X, \forall\, \lambda \in A$. The a mapping $<.> : SE(\check{X}) \times SE(\check{X}) \to \mathbb{C}(A)$ is said to be a soft inner product on the soft vector space $\check{X}$ if $<.>$ satisfies the following conditions:

(I1). $< \tilde{x}, \tilde{x} > \geq \bar{0}$, for all $\tilde{x} \mathrel{\tilde{\in}} \check{X}$ and $< \tilde{x}, \tilde{x} > = \bar{0}$ if and only if $\tilde{x} = \Phi$;

(I2). $< \tilde{x}, \tilde{y} > = \overline{< \tilde{y}, \tilde{x} >}$, where bar denote the complex conjugate of soft complex numbers;

(I3) $< \tilde{\alpha}.\tilde{x}, \tilde{y} > = \tilde{\alpha} < \tilde{x}, \tilde{y} >$, for all $\tilde{x}, \tilde{y} \tilde{\in} \breve{X}$ and for every soft scalar $\tilde{\alpha}$;

(I4) For all $\tilde{x}, \tilde{y} \tilde{\in} \breve{X}$, $< \tilde{x} + \tilde{y}, \tilde{z} > = < \tilde{x}, \tilde{z} > + < \tilde{y}, \tilde{z} >$.

The soft vector space $\breve{X}$ with a soft inner product $<.>$ on $\breve{X}$ is said to be a soft inner product space and is denoted by $(\breve{X}, <.>, A)$ or $(\breve{X}, <.>)$. (I1), (I2) , (I3) and (I4) are said to be soft inner product axioms.

Proposition 2.18 (see [8]): Let $\{ <.>_\lambda : \lambda \in A \}$ be a family of crisp inner products on a crisp vector space X. Then the mapping $<.>: SE(\breve{X}) \times SE(\breve{X}) \rightarrow \mathbb{C}(A)$ by $< \tilde{x}, \tilde{y} > (\lambda) = < \tilde{x}(\lambda), \tilde{y}(\lambda) >_\lambda, \forall \lambda \in A, \ \tilde{x}, \tilde{y} \tilde{\in} \breve{X}$ is a soft inner product on the soft linear space $\breve{X}$.

Theorem 2.19 (see [8]): If a soft inner product $<.>$ satisfies the condition (I5) , and if for each $\lambda \in A$, $<.>_\lambda : X \times X \rightarrow \mathbb{C}$ be a mapping such that for all $(\xi, \eta) \in X \times X$, $< \xi, \eta >_\lambda = < \tilde{x}, \tilde{y} > (\lambda)$, where $\tilde{x}, \tilde{y} \tilde{\in} \breve{X}$ such that $\tilde{x}(\lambda) = \xi$, $\tilde{y}(\lambda) = \eta$. Then for each $\lambda \in A$, $<.>_\lambda$ is an inner product on X.

(I5) For each $(\xi, \eta) \in X \times X$ and $\in A$, $\{ < \tilde{x}, \tilde{y} > (\lambda) : \tilde{x}, \tilde{y} \tilde{\in} \breve{X}$ such that $\tilde{x}(\lambda) = \xi$, $\tilde{y}(\lambda) = \eta \}$ is a singleton set.

Definition 2.20 (see [7]): Let G be a soft vector space of V over K. Let $\widetilde{\alpha_1}, \widetilde{\alpha_2}, \dots, \widetilde{\alpha_n} \in G$. A soft vector $\tilde{\beta}$ is said to be a linear combination of the soft vectors $\widetilde{\alpha_1}, \widetilde{\alpha_2}, \dots, \widetilde{\alpha_n}$ if $\tilde{\beta}$ can be expressed as a $\tilde{\beta} = \widetilde{c_1}.\widetilde{\alpha_1} + \widetilde{c_2}\,\widetilde{\alpha_2} + \dots + \widetilde{c_n}\widetilde{\alpha_n}$, for some soft scalars $\widetilde{c_1}, \widetilde{c_2}, \dots, \widetilde{c_n}$.

Definition 2.21 (see [7]): A finite set of soft vectors $\{ \widetilde{\alpha_1}, \widetilde{\alpha_2}, \dots, \widetilde{\alpha_n} \}$ of soft vector space G is said to be linearly independent in G if there exists soft scalars $\widetilde{c_1}, \widetilde{c_2}, \dots, \widetilde{c_n}$ not all $\bar{0}$ such that $\widetilde{c_1}.\widetilde{\alpha_1} + \widetilde{c_2}\,\widetilde{\alpha_2} + \dots + \widetilde{c_n}\widetilde{\alpha_n} = \Phi$. An arbitrary set S of soft vectors of G is said to be linearly dependent in G if there exists a finite subset of S which is linearly dependent in G.

Definition 2.22 (see [20]): Let $\breve{X}$ be a soft linear space. A set S of soft vectors in $\breve{X}$ is said to be a basis of $\breve{X}$ if S is linearly independent and S generates $\breve{X}$ i.e. any soft element of $\breve{X}$ can be expressed as a linear combination of those linearly independent soft vectors.

Definition 2.23 (see [20]): If the basis set S of soft vectors is finite then $\breve{X}$ is said to be a finite dimensional soft linear space and the number of soft vectors of the basis is called the dimension of the soft linear space $\breve{X}$.

(I2). $< \tilde{x}, \tilde{y} > = \overline{< \tilde{y}, \tilde{x} >}$, where bar denote the complex conjugate of soft complex numbers;

(I3) $< \tilde{\alpha}.\tilde{x}, \tilde{y} > = \tilde{\alpha} < \tilde{x}, \tilde{y} >$, for all $\tilde{x}, \tilde{y} \tilde{\in} \breve{X}$ and for every soft scalar $\tilde{\alpha}$;

(I4) For all $\tilde{x}, \tilde{y} \tilde{\in} \breve{X}$, $< \tilde{x} + \tilde{y}, \tilde{z} > = < \tilde{x}, \tilde{z} > + < \tilde{y}, \tilde{z} >$.

The soft vector space $\breve{X}$ with a soft inner product $<.>$ on $\breve{X}$ is said to be a soft inner product space and is denoted by $(\breve{X}, <.>, A)$ or $(\breve{X}, <.>)$. (I1), (I2) , (I3) and (I4) are said to be soft inner product axioms.

Proposition 2.18 (see [8]): Let $\{ <.>_\lambda : \lambda \in A \}$ be a family of crisp inner products on a crisp vector space X. Then the mapping $<.>: SE(\breve{X}) \times SE(\breve{X}) \to \mathbb{C}(A)$ by $< \tilde{x}, \tilde{y} > (\lambda) = < \tilde{x}(\lambda), \tilde{y}(\lambda) >_\lambda, \forall \lambda \in A,\ \tilde{x}, \tilde{y} \tilde{\in} \breve{X}$ is a soft inner product on the soft linear space $\breve{X}$.

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(I5) For each $(\xi, \eta) \in X \times X$ and $\in A$, $\{ < \tilde{x}, \tilde{y} > (\lambda) : \tilde{x}, \tilde{y} \tilde{\in} \breve{X}$ such that $\tilde{x}(\lambda) = \xi$, $\tilde{y}(\lambda) = \eta \}$ is a singleton set.

Definition 2.20 (see [7]): Let G be a soft vector space of V over K. Let $\widetilde{\alpha_1}, \widetilde{\alpha_2}, \dots, \widetilde{\alpha_n} \in G$. A soft vector $\tilde{\beta}$ is said to be a linear combination of the soft vectors $\widetilde{\alpha_1}, \widetilde{\alpha_2}, \dots, \widetilde{\alpha_n}$ if $\tilde{\beta}$ can be expressed as a $\tilde{\beta} = \tilde{c_1}.\widetilde{\alpha_1} + \tilde{c_2}\,\widetilde{\alpha_2} + \dots + \widetilde{c_n}\widetilde{\alpha_n}$, for some soft scalars $\tilde{c_1}, \tilde{c_2}, \dots, \widetilde{c_n}$.

Definition 2.21 (see [7]): A finite set of soft vectors $\{ \widetilde{\alpha_1}, \widetilde{\alpha_2}, \dots, \widetilde{\alpha_n} \}$ of soft vector space G is said to be linearly independent in G if there exists soft scalars $\tilde{c_1}, \tilde{c_2}, \dots, \widetilde{c_n}$ not all $\bar{0}$ such that $\tilde{c_1}.\widetilde{\alpha_1} + \tilde{c_2}\,\widetilde{\alpha_2} + \dots + \widetilde{c_n}\widetilde{\alpha_n} = \Phi$. An arbitrary set S of soft vectors of G is said to be linearly dependent in G if there exists a finite subset of S which is linearly dependent in G.

Definition 2.22 (see [20]): Let $\breve{X}$ be a soft linear space. A set S of soft vectors in $\breve{X}$ is said to be a basis of $\breve{X}$ if S is linearly independent and S generates $\breve{X}$ i.e. any soft element of $\breve{X}$ can be expressed as a linear combination of those linearly independent soft vectors.

Definition 2.23 (see [20]): If the basis set S of soft vectors is finite then $\breve{X}$ is said to be a finite dimensional soft linear space and the number of soft vectors of the basis is called the dimension of the soft linear space $\breve{X}$.

Definition 2.24 (See [20]) **:** Let $\tilde{X}$ be soft inner product space which satisfies (I5). Then $\tilde{X}$ is said to be complete if it is complete with respect to the soft metric induced by soft inner product. A complete soft inner product space is said to be a soft Hilbert space and we denote the soft Hilbert space by $\tilde{H}$.

3. Soft Hilbert Space

Let X be a vector space over a field $\mathbb{C}$ of complex number, X is our initial universal set and A be a non-empty set of parameters. Let $\tilde{X}$ be the absolute soft vector space i.e. $\tilde{X}(\lambda) = X$. for all $\lambda \in A$. We use the notation $\tilde{x}, \tilde{y}, \tilde{z}$ to denote soft vectors of a soft vector space and $\tilde{r}, \tilde{s}, \tilde{t}$ to denote soft real numbers whereas $\bar{r}, \bar{s}, \bar{t}$ will denote a particular type of soft real numbers such that $\bar{r}(\lambda) = r$, for all $\lambda \in A$. For example $\bar{0}$ is the soft real number where $\bar{0}(\lambda) = 0$, for all $\lambda \in A$. In this section we have given the notion of orthogonality, orthonormality in soft Hilbert space and also discuss some properties of this space.

Definition 3.1: Let $\tilde{L}$ be a non-null soft subset of the soft Hilbert space $\tilde{H}$ such that $\tilde{L}(\lambda) \neq \phi, \forall \lambda \in A$. Two soft vectors $\tilde{x}, \tilde{y}$ of $\tilde{H}$ are said to be orthogonal if $< \tilde{x}, \tilde{y} > = \bar{0}$. In symbol, we write $\tilde{x} \perp \tilde{y}$. If $\tilde{x}$ is orthogonal to every soft vectors of $\tilde{L}$ then we say that $\tilde{x}$ is orthogonal to $\tilde{L}$ and we write $\tilde{x} \perp \tilde{L}$.

Definition 3.2: Let $\tilde{H}$ be a Hilbert space. Then a collection $\mathcal{B}$ of soft vectors of $\tilde{H}$ is said to be orthonormal if for all $\tilde{x}, \tilde{y} \in \mathcal{B}$

$$< \tilde{x}, \tilde{y} > = \begin{cases} \bar{0} \ ; if\ \tilde{x} \neq \tilde{y} \\ \bar{1} ;\ if\ \tilde{x} = \tilde{y} \end{cases}$$

If the soft set $\mathcal{B}$ contains only a countable number of soft vectors then we can arrange it in a sequence of soft vectors and call it an orthonormal sequence.

Definition 3.3: Let $\tilde{H}$ be a Hilbert space. Then a basis S of soft vectors of $\tilde{H}$ is soft orthonormal basis if the set S is orthonormal i.e. for all $\tilde{x}, \tilde{y} \in S$,

$$< \tilde{x}, \tilde{y} > = \begin{cases} \bar{0} \ ; if\ \tilde{x} \neq \tilde{y} \\ \bar{1} ;\ if\ \tilde{x} = \tilde{y} \end{cases}$$

Theorem 3.4: Every soft Hilbert space of finite dimension possesses an orthonormal basis.

Proof: Let $\{\widetilde{x_1}, \widetilde{x_2}, \ldots, \widetilde{x_n}\}$ be a basis of the finite dimensional soft Hilbert space $\widetilde{H}$. An orthogonal basis of $\widetilde{H}$ will be obtained by the Gram-Schmidt process of orthogonalisation. Since the basis vectors are non-null, we select one of them , say $\widetilde{x_1}$, and consider as the first member of the new basis. For convenience we rename it as $\widetilde{y_1}$ i.e. $\widetilde{y_1} = \widetilde{x_1}$.

Now let $\widetilde{y_2} = \widetilde{x_2} - \widetilde{c_1}\widetilde{y_1}$, where $\widetilde{c_1}\widetilde{y_1}$ is the projection of $\widetilde{x_2}$ upon $\widetilde{y_1}$.

Then $< \widetilde{y_2}, \widetilde{y_1} > = < \widetilde{x_2} - \widetilde{c_1}\widetilde{y_1}, \widetilde{y_1} > = < \widetilde{x_2}, \widetilde{y_1} > - \widetilde{c_1} < \widetilde{y_1}, \widetilde{y_1} >$

$$= < \widetilde{x_2}, \widetilde{y_1} > - \frac{<\widetilde{x_2}, \widetilde{y_1}>}{<\widetilde{y_1}, \widetilde{y_1}>} < \widetilde{y_1}, \widetilde{y_1} > = \bar{0}.$$

This gives $\widetilde{y_2}$ is orthogonal to $\widetilde{x_1}$ and $\{\widetilde{y_1}, \widetilde{y_2}\} = L\{\widetilde{y_1}, \widetilde{x_2}\} = L\{\widetilde{x_1}, \widetilde{x_2}\}$, where $L\{\widetilde{y_1}, \widetilde{y_2}\}$ denote the linear span of the soft vectors $\widetilde{y_1}$ and $\widetilde{y_2}$.

$$\widetilde{y_2} = \widetilde{x_2} - \frac{< \widetilde{x_2}, \widetilde{y_1} >}{< \widetilde{y_1}, \widetilde{y_1} >} \widetilde{y_1}$$

Now clearly $\widetilde{x_3} \notin L\{\widetilde{y_1}, \widetilde{y_2}\}$. Let $\widetilde{y_3} = \widetilde{x_3} - \widetilde{d_1}\,\widetilde{y_1} - \widetilde{d_2}\,\widetilde{y_2}$, where $\widetilde{d_1}\,\widetilde{y_1}, \widetilde{d_2}\,\widetilde{y_2}$ are the projections of $\widetilde{x_3}$ upon $\widetilde{y_1}$, $\widetilde{y_2}$ respectively.

Then, $< \widetilde{y_3}, \widetilde{y_2} > = < \widetilde{x_3} - \widetilde{d_1}\,\widetilde{y_1} - \widetilde{d_2}\,\widetilde{y_2}, \widetilde{y_2} >$

$$= < \widetilde{x_3}, \widetilde{y_2} > - \widetilde{d_1} < \widetilde{y_1}, \widetilde{y_2} > - \widetilde{d_2} < \widetilde{y_2}, \widetilde{y_2} >$$

$$= < \widetilde{x_3}, \widetilde{y_2} > - \widetilde{d_2} < \widetilde{y_2}, \widetilde{y_2} >, \text{ since } < \widetilde{y_1}, \widetilde{y_2} > = \bar{0}$$

$$= < \widetilde{x_3}, \widetilde{y_2} > - \frac{<\widetilde{x_3}, \widetilde{y_2}>}{<\widetilde{y_2}, \widetilde{y_2}>} < \widetilde{y_2}, \widetilde{y_2} >$$

$$= < \widetilde{x_3}, \widetilde{y_2} > - < \widetilde{x_3}, \widetilde{y_2} > = \bar{0}$$

In a similar way one can check that $< \widetilde{y_3}, \widetilde{y_1} > = \bar{0}$.

This gives $\widetilde{y_3}$ is orthogonal to $\widetilde{y_2}, \widetilde{y_1}$ and $L\{\widetilde{y_1}, \widetilde{y_2}, \widetilde{y_3}\} = L\{\widetilde{y_1}, \widetilde{y_2}, \widetilde{x_3}\} = L\{\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_3}\}$.

$$\widetilde{y_3} = \widetilde{x_3} - \frac{< \widetilde{x_3}, \widetilde{y_1} >}{< \widetilde{y_1}, \widetilde{y_1} >}\widetilde{y_1} - \frac{< \widetilde{x_3}, \widetilde{y_2} >}{< \widetilde{y_2}, \widetilde{y_2} >}\widetilde{y_2}$$

Now $\widetilde{x_4} \notin L\{\widetilde{y_1}, \widetilde{y_2}, \widetilde{y_3}\}$. Let $\widetilde{y_4} = \widetilde{x_4} - \widetilde{r_1}\,\widetilde{y_1} - \widetilde{r_2}\,\widetilde{y_2} - \widetilde{r_3}\,\widetilde{y_3}$, where $\widetilde{r_1}\,\widetilde{y_1}, \widetilde{r_2}\,\widetilde{y_2}, \widetilde{r_3}\,\widetilde{y_3}$ are the projections of $\widetilde{x_4}$ upon $\widetilde{y_1}, \widetilde{y_2}, \widetilde{y_3}$ respectively.

As proceeding above we can show that $\widetilde{y_4}$ is orthogonal to $\widetilde{y_1}, \widetilde{y_2}, \widetilde{y_3}$ and $L\{\widetilde{y_1}, \widetilde{y_2}, \widetilde{y_3}, \widetilde{y_4}\} = L\{\widetilde{y_1}, \widetilde{y_2}, \widetilde{y_3}, \widetilde{x_4}\} = L\{\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_3}, \widetilde{x_4}\}$.

$$\widetilde{y_4} = \widetilde{x_4} - \frac{<\widetilde{x_4}, \widetilde{y_1}>}{<\widetilde{y_1}, \widetilde{y_1}>}\widetilde{y_1} - \frac{<\widetilde{x_4}, \widetilde{y_2}>}{<\widetilde{y_2}, \widetilde{y_2}>}\widetilde{y_2} - \frac{<\widetilde{x_4}, \widetilde{y_1}>}{<\widetilde{y_1}, \widetilde{y_1}>}\widetilde{y_3}$$

This process concludes after a finite number of steps because, at each step, a vector from the original basis is replaced with a vector from the targeted orthogonal basis. Thus we have,

$$\widetilde{y_n} = \widetilde{x_n} - \frac{<\widetilde{x_n}, \widetilde{y_1}>}{<\widetilde{y_1}, \widetilde{y_1}>}\widetilde{y_1} - \frac{<\widetilde{x_n}, \widetilde{y_2}>}{<\widetilde{y_2}, \widetilde{y_2}>}\widetilde{y_2} - \cdots - \frac{<\widetilde{x_n}, \widetilde{y_{n-1}}>}{<\widetilde{y_{n-1}}, \widetilde{y_{n-1}}>}\widetilde{y_{n-1}}$$

and $\{\widetilde{y_1}, \widetilde{y_2}, \dots, \widetilde{y_n}\}$ is an orthogonal basis of the soft Hilbert space $\tilde{H}$. This completes the proof.

Example 3.5 (Soft Hilbert Space)**:** For $p = 2$, the soft vector space (F, A) of $\mathbb{R}^I$ over $\mathbb{R}$, define as $F(\lambda) = l^p \subset \mathbb{R}^I$, $\forall \lambda \in A$ with the soft l^2 norm is a soft Hilbert space and denoted by $\widetilde{l^2}$, where the soft l^2 norm is defined as, $\|.\|: SE(\widetilde{l^2}) \to \mathbb{R}(A)^*$ by, $\|\tilde{x}\|(\lambda) = \|\tilde{x}(\lambda)\|_2 = (\sum_{i\in I}|\xi_i(\lambda)|^2)^{\frac{1}{2}}$.

Definition 3.7: Let $\tilde{\beta}$ be a fixed non-null soft vector in a Hilbert space $\tilde{H}$. Then for a non-null soft vector $\tilde{\alpha}$ in $\tilde{H}$ there exists a unique soft real number $\tilde{c}$ such that $\tilde{\alpha} - \tilde{c}\tilde{\beta}$ is orthogonal to $\tilde{\beta}$. $\tilde{c}$ is determined by the relation $< \tilde{\alpha} - \tilde{c}\tilde{\beta}, \tilde{\beta} > = 0$. Therefore $< \tilde{\alpha}, \tilde{\beta} > = \tilde{c} < \tilde{\beta}, \tilde{\beta} >$, giving $\tilde{c} = \frac{<\tilde{\alpha},\tilde{\beta}>}{<\tilde{\beta},\tilde{\beta}>}$.

$\tilde{c}$ is said to be the scalar component of $\tilde{\alpha}$ along $\tilde{\beta}$ and $\tilde{c}\tilde{\beta}$ is said to be the projection of $\tilde{\alpha}$ upon $\tilde{\beta}$.

Theorem 3.8 : For $\tilde{x}, \tilde{y} \tilde{\in} \widetilde{l^2}$,

$$\|\widetilde{\tilde{x} + \tilde{y}}\|_2^2 + \|\widetilde{\tilde{x} - \tilde{y}}\|_2^2 = 2\left(\|\widetilde{\tilde{x}}\|_2^2 + \|\widetilde{\tilde{y}}\|_2^2\right)$$

Proof: Let $\tilde{x}, \tilde{y} \tilde{\in} \widetilde{l^2}$. Then we have $\tilde{x}(\lambda) \in l^2$ and $\tilde{y}(\lambda) \in l^2$. Now since l^2 is a Hilbert space we have,

$$\|\tilde{x}(\lambda) + \tilde{y}(\lambda)\|_2^2 + \|\tilde{x}(\lambda) - \tilde{y}(\lambda)\|_2^2$$

$$= < \tilde{x}(\lambda) + \tilde{y}(\lambda), \tilde{x}(\lambda) + \tilde{y}(\lambda) > + < \tilde{x}(\lambda) - \tilde{y}(\lambda), \tilde{x}(\lambda) - \tilde{y}(\lambda) >$$

$$= < \tilde{x}(\lambda), \tilde{x}(\lambda) > + < \tilde{x}(\lambda), \tilde{y}(\lambda) > + < \tilde{y}(\lambda), \tilde{x}(\lambda) > + < \tilde{y}(\lambda), \tilde{y}(\lambda) > + < \tilde{x}(\lambda), \tilde{x}(\lambda) > - < \tilde{x}(\lambda), \tilde{y}(\lambda) > - < \tilde{y}(\lambda), \tilde{x}(\lambda) > + < \tilde{y}(\lambda), \tilde{y}(\lambda) >$$

$= 2\,(< \tilde{x}(\lambda), \tilde{x}(\lambda) > + < \tilde{y}(\lambda), \tilde{y}(\lambda) >) = 2\,(\|\tilde{x}(\lambda)\|^2 + \|\tilde{y}(\lambda)\|^2)$, for all $\lambda \in A$.

Therefore, $\|\widetilde{\tilde{x} + \tilde{y}}\|_2^2 + \|\widetilde{\tilde{x} - \tilde{y}}\|_2^2 = 2\left(\|\widetilde{\tilde{x}}\|_2^2 + \|\widetilde{\tilde{y}}\|_2^2\right)$. This gives the parallelogram law for soft $\widetilde{l^2}$ space.

Remark 3.6: Since the only l^p norm which satisfies the parallelogram law in normed linear space is for $p = 2$, so by using the result that a 'Banach space' which satisfies the 'parallelogram law' is a 'Hilbert space', we can say that the only soft l^p space which can be consider as a Soft Hilbert space is soft l^2 space.

Conclusion:

In this present work we have analysed some significant characteristic like orthogonality, orthonormality in soft Hilbert space are investigated. Also we discussed the soft l^2 space as an example of soft Hilbert space and some inequalities on this space have been given. In addition, we established that every soft Hilbert space admits an orthonormal basis, constructed via the Gram–Schmidt orthogonalization process on a soft inner product space.

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Conflicts of interest
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References:
1. H. Aktas and N. Cagman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2226-2735.
2. M. I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57 (2009) 1547-1553.
3. D. Chen, The parametrization reduction of soft sets and its applications, Comput. Math. Appl. 49 (2005) 757-763.
4. S. Das and S. K. Samanta, Soft real sets, soft real numbers and their properties, J. Fuzzy Math. 20(3) (2012) 551-576.
5. S. Das and S. K. Samanta, On soft complex sets and soft complex numbers, J. Fuzzy Math. 21(1) (2013) 195-216.
6. S. Das and S. K. Samanta, On soft metric spaces, J. Fuzzy Math. Accepted.
7. S. Das, Pinaki Majumdar and S. K. Samanta, On soft linear spaces and soft Normed linear spaces, Communicated to Ukrainian Math. J.
8. S. Das and S. K. Samanta , On soft inner product spaces , Ann. Fuzzy Math. Inform. 6 (2013), No. 1, 151-170.
9. S. Das and S. K. Samanta , Soft linear functionals in soft normed linear spaces , Ann. Fuzzy Math. Inform. 7 (2014), No. 4, 629–651
10. F. Feng, Y. B. Jun and X. Zhao, Soft semirings, Comput. Math. Appl. 56 (2008) 2621-2628.
11. F. Feng, C. X. Li, B. Davvaz and M. I. Ali, Soft sets combined with fuzzy sets and rough sets: a tentative approach, Soft Computing 14 (2010) 8999-9911.
12. Y. B. Jun, Soft BCK/BCI-algebras, Comput. Math. Appl. 56 (2008) 1408-1413.
13. Z. Kong, L. Gao, L. Wong and S. Li, The normal parameter reduction of soft sets and its algorithm, Comput. Math. Appl. 56 (2008) 3029-3037.
14. P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. $45(2003)\ 555-562$.
15. P. K. Maji, R. Biswas and A. R. Roy, An application of soft sets in a decision making problem, Comput. Math. Appl. 44(2002) 1077-1083.
16. P. Majumdar and S. K. Samanta, On soft mappings, Comput. Math. Appl. 60 (2010) 2666-2672.
17. D. Molodtsov, Soft set theory first results, Comput. Math. Appl. $37(1999)\ 19-31$.
18. D. Pie and D. Miao, From soft sets to information systems, Granular Computing, 2005, IEEE International Conference, Volume 2, 617-622.
19. M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (2011) 1786-1799.
20. Y. Zou and Z. Xiao, Data analysis approaches of soft sets under incomplete information, Knowledge-Based Systems 21 (2008) 941-945.
21. Indubaran Mandal, Proceedings of the Two-Day Conference on Recent Trends in Mathematics and its Applications, Red'shine Publication , ISBN: 978-93-5879-746-6 (2025) , 177-188